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# Some geometric aspects of CP$^{N-1}$ maps

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**Abstract.** In this talk we introduce a Weierstrass-like system of equations corresponding to $CP^{N-1}$ fields in two dimensions. Then using this representation we introduce a vector $\boldsymbol{r}$ in $R^{N^2-1}$ and treating this vector as the radius vector of a surface immersed in $R^{N^2-1}$ we discuss to what extent the associated metric describes the geometry of the $CP^{N-1}$ maps. We show that for the holomorphic maps – the correspondence is exact; while for the more general fields we have to go beyond the Weierstrass system and add extra terms.

**PACS.** 02.40.-K Geometry, differential geometry, and topology – 02.30.Jr Partial differential equations – 11.10.Lm Nonlinear or nonlocal theories and models

## 1 Introduction

In this talk I want to discuss two interesting ideas, and then relate them to each other. Moreover, I will then show how these ideas can be further generalised.

The work described in this talk has been done in collaboration with M. Grundland. The details of the work will be given elsewhere [1].

### 1.1 Harmonic maps [2]

The first topic involves $CP^1$ harmonic maps.

These are maps

$$S^2 \to CP^1 \sim S^2 \qquad (1)$$

given by the stationary points of the energy

$$\mathcal{L} = \int L \, \mathrm{d}x \, \mathrm{d}y, \qquad (2)$$

where

$$L = \frac{1}{4}(D_\mu z)^\dagger \cdot D_\mu z, \qquad (3)$$

and where, in the general case of $CP^{N-1}$, $z$ is a vector field of $N$ components, $z = (z^1, ..., z^N)$, which satisfies

$$z^\dagger \cdot z = 1. \qquad (4)$$

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The differential operator $D_\mu$ acts on $\psi\colon S^2 \to CP^{N-1}$ according to the formula:

$$D_\mu \psi = \partial_\mu \psi - \psi \,(z^\dagger \cdot \partial_\mu z). \qquad (5)$$

Here $\mu = 1, 2$, of course, and denotes the space coordinates $x$ and $y$. In the $CP^1$ case we can introduce a complex field $W$

$$z = \frac{(1, W)}{\sqrt{1 + |W|^2}}. \qquad (6)$$

Then, the Euler Lagrange equations describing harmonic maps are given by

$$\partial \bar\partial W - 2\bar{W}\frac{\partial W \bar\partial W}{|W|^2 + 1} = 0 \qquad (7)$$

where $W = W(\zeta, \bar\zeta)$ and

$$\partial = \frac{\partial}{\partial(x + \mathrm{i}y)} = \frac{\partial}{\partial \zeta}, \qquad \bar\partial = \frac{\partial}{\partial \bar\zeta}. \qquad (8)$$

### 1.2 Weierstrass system [3]

Here we consider 2 complex functions $\psi = \psi(\zeta, \bar\zeta)$ and $\phi = \phi(\zeta, \bar\zeta)$, which satisfy

$$\partial \psi = p\,\phi, \qquad \bar\partial \phi = -p\,\psi, \qquad p = |\phi|^2 + |\psi|^2. \quad (9)$$

Note that we have not specified $\bar\partial\psi$, nor $\partial\phi$.

A natural question then arises. Are these two problems related? Obviously the answer is YES. To see this put

$$W = \frac{\psi}{\bar\phi} \qquad (10)$$

and

$$\psi = W \frac{(\bar{\partial}\bar{W})^{\frac{1}{2}}}{1+|W|^2}, \qquad \phi = \frac{(\partial W)^{\frac{1}{2}}}{1+|W|^2}, \qquad (11)$$

and so find that (7) and (9) are equivalent.

**Moreover**, we can introduce 3 real quantities

$$X_1 = \mathrm{i} \int_\gamma [\bar{\psi}^2 + \phi^2]\mathrm{d}\zeta - [\psi^2 + \bar{\phi}^2]\mathrm{d}\bar{\zeta},$$

$$X_2 = \int_\gamma [\bar{\psi}^2 - \phi^2]\mathrm{d}\zeta + [\psi^2 - \bar{\phi}^2]\mathrm{d}\bar{\zeta},$$

$$X_3 = -2 \int_\gamma \bar{\psi}\phi\mathrm{d}\zeta + \psi\bar{\phi}\mathrm{d}\bar{\zeta}, \qquad (12)$$

where $\gamma$ is any curve from a fixed point to $\zeta$.

Then, it is easy to show that if $\psi$ and $\phi$ satisfy (9) then $X_i$ do not depend on the details of the curve $\gamma$ but only on its endpoints.

Furthermore, if we treat $X_i$ as components of a vector $\boldsymbol{r} = (X_1, X_2, X_3)$ and introduce the metric

$$g_{\zeta\zeta} = (\partial\boldsymbol{r}, \partial\boldsymbol{r}), \qquad g_{\bar{\zeta}\bar{\zeta}} = (\bar{\partial}\boldsymbol{r}, \bar{\partial}\boldsymbol{r}), \qquad g_{\zeta\bar{\zeta}} = (\partial\boldsymbol{r}, \bar{\partial}\boldsymbol{r}) \qquad (13)$$

we find that, for fields which solve (7) on $S^2$, only $g_{\zeta\bar{\zeta}}$ is non-zero and is given by

$$g_{\zeta\bar{\zeta}} = \frac{|\partial W|^2}{(1+|W|^2)^2} = |Dz|^2, \qquad (14)$$

where $D$ denotes the $D_\mu$ derivative (5) but evaluated with respect to $\zeta$. Note that (14) is a term in the general expression for the energy of the $CP^1$ map. However, as all harmonic maps on $S^2$ satisfy $W = W(\zeta)$ [2], $g_{\zeta\bar{\zeta}}$ is the energy[1].

Can we generalise this to $CP^{N-1}$?

## 2 $CP^{N-1}$ case

### 2.1 General considerations [2]

In the $CP^{N-1}$ case we can put

$$z_i = \frac{f_i}{|f|} \qquad i = 1, ...N, \qquad (15)$$

and introduce $W_k = \frac{f_k}{f_1}$. The energy is still given by (3) and the Euler Lagrange equations become

$$\left(1 - \frac{ff^\dagger}{|f|^2}\right)\left[\partial\bar{\partial}f - \partial f \frac{(f^\dagger \cdot \bar{\partial}f)}{|f|^2} - \bar{\partial}f \frac{(f^\dagger \cdot \partial f)}{|f|^2}\right] = 0. \qquad (16)$$

Holomorphic solutions imply $\bar{\partial}f = 0$ but, for $N > 2$ there exist nonholomorphic solutions of (16).

To construct the generalised Weierstrass system we note that (16) can be rewritten as

$$[\partial\bar{\partial}P, P] = 0 \qquad \text{where} \quad P = \frac{1}{A}f f^\dagger, \qquad (17)$$

where $A = f^\dagger f$. Thus we have

$$\partial K + \bar{\partial}M = 0, \quad \text{with} \quad K = [\bar{\partial}P, P], \ M = [\partial P, P] \qquad (18)$$

The explicit form of matrices $K$ and $M$ is given by[2]

$$K_{ij} = \frac{1}{A^2}[\bar{f}_k f_k \bar{\partial}f_i \bar{f}_j - \bar{f}_k f_k f_i \bar{\partial}\bar{f}_j$$
$$+ f_i \bar{f}_j \bar{\partial}\bar{f}_k f_k - f_i \bar{f}_j \bar{f}_k \bar{\partial}f_k], \qquad (19)$$

$$M_{ij} = \frac{1}{A^2}[\bar{f}_k f_k \partial f_i \bar{f}_j - \bar{f}_k f_k f_i \partial\bar{f}_j$$
$$+ f_i \bar{f}_j \partial\bar{f}_k f_k - f_i \bar{f}_j \bar{f}_k \partial f_k], \qquad (20)$$

which we next rewrite as

$$K_{ij} = \bar{f}_j \bar{\Phi}_i^2 - f_i \bar{\varphi}_j^2, \qquad M_{ij} = \bar{f}_j \varphi_i^2 - f_i \Phi_j^2, \qquad (21)$$

where we have defined

$$\varphi_i^2 = \frac{1}{A^2} \bar{f}_k F_{ki}, \qquad \Phi_i^2 = \frac{1}{A^2} f_k \tilde{\bar{F}}_{ki}, \qquad (22)$$

and

$$F_{ij} = f_i \partial f_j - f_j \partial f_i, \qquad \tilde{F}_{ij} = f_i \bar{\partial}f_j - f_j \bar{\partial}f_i. \qquad (23)$$

Note that we have two constraints

$$\bar{f}_k \varphi_k^2 = 0, \qquad f_k \Phi_k^2 = 0, \qquad (24)$$

which allows us to consider as independent only $\varphi_2, ..\varphi_N$. At the same time, as we have said before, we can set, say, $f_1 = 1$ and so we end up with

$$\varphi_i^2 = \frac{1}{A^2}\left[(1 + f_k\bar{f}_k)\partial f_i - f_i(\bar{f}_k \partial f_k)\right], \qquad (25)$$

where all the sums over repeated indices run over $k = 2, ...N$.

Note that the $CP^{N-1}$ system has more conserved quantities. In fact, we can drop the $\Phi$ terms in the expressions for $K$ and $M$ in (21) and we still have our conservation laws; namely, we can define

$$K'_{ij} = -f_i \bar{\varphi}_j^2, \qquad M'_{ij} = \varphi_i^2 \bar{f}_j \qquad (26)$$

and then note that we still have

$$\partial K' + \bar{\partial}M' = 0. \qquad (27)$$

This is easy to check the validity of (27) by using (16).

## 2.2 Weierstrass system [1]

As our generalised Weierstrass variables we now take

$$\varphi_i \quad \text{and} \quad \psi_i = f_i\bar{\varphi}_i, \quad i = 2,...N. \qquad (28)$$

Then the generalised Weierstrass system is given by the equations

$$\bar{\partial}\varphi_i = -\varphi_i A (\psi \cdot \bar{\varphi}), \quad \partial\psi_i = \varphi_i A |\varphi_i|^2 \quad \text{no summation}, \qquad (29)$$

where

$$A = 1 + \sum_{k=1}^{N} \frac{|\psi_k|^2}{|\varphi_k|^2}. \qquad (30)$$

It can be shown that these equations are equivalent to the $CP^{N-1}$ equations (16) or (17).

Moreover, we can also construct $N^2 - 1$ real quantities $X_i$, which we can be treated as components of a vector $\boldsymbol{r}$ in $N^2 - 1$ dimensions.

To construct them we take various entries of our matrices $K'$ and $M'$.

Explicitly, the diagonal terms give us ($l = 1,...,N$ no summation)

$$X_{ll} = \int_\gamma \bar{f}_l\varphi_l^2 \mathrm{d}\zeta + \int_\gamma f_l\bar{\varphi}_l^2 \mathrm{d}\bar{\zeta} = \int_\gamma \bar{\psi}_l\varphi_l \mathrm{d}\zeta + \int_\gamma \psi_l\bar{\varphi}_l \mathrm{d}\bar{\zeta} \qquad (31)$$

which satisfy $\sum_l X_{ll} = 0$. The off-diagonal terms give us

$$X_{lk} = \frac{1}{2}\left[\int_\gamma \left(\bar{\psi}_l\frac{\varphi_k^2}{\bar{\varphi}_l} + \bar{\psi}_k\frac{\varphi_l^2}{\bar{\varphi}_k}\right)\mathrm{d}\zeta + \int_\gamma \left(\psi_l\frac{\bar{\varphi}_k^2}{\varphi_l} + \psi_k\frac{\bar{\varphi}_l^2}{\varphi_k}\right)\mathrm{d}\bar{\zeta}\right] \qquad (32)$$

$$Y_{lk} = \frac{\mathrm{i}}{2}\left[\int_\gamma \left(\bar{\psi}_l\frac{\varphi_k^2}{\bar{\varphi}_l} - \bar{\psi}_k\frac{\varphi_l^2}{\bar{\varphi}_k}\right)\mathrm{d}\zeta + \int_\gamma \left(\psi_l\frac{\bar{\varphi}_k^2}{\varphi_l} - \psi_k\frac{\bar{\varphi}_l^2}{\varphi_k}\right)\mathrm{d}\bar{\zeta}\right]. \qquad (33)$$

In our expressions we take all $l, k = 1,...,N$ and for $k = 1$ and $l = 1$ we use our constraints to rewrite all our expressions in terms of independent $\varphi_i$ and $\psi_i$, $i = 2,...,N$. As in the $CP^1$ case there is no dependence on the contour of integration except through its endpoints.

Moreover, when we calculate the metric – we find, like before, that the only nonvanishing entry is

$$g_{\zeta\bar{\zeta}} = |Dz|^2. \qquad (34)$$

On the other hand, if we go beyond the generalised Weierstrass system and consider also $\Phi$ terms in $K$ and $M$ and define our $X_i$ variables in terms of these matrices and not $K'$ and $M'$ we find that the metric now becomes

$$g_{\zeta\bar{\zeta}} = |Dz|^2 + |\bar{D}z|^2 \qquad (35)$$

*i.e.* is the total energy.

## 3 Conclusions

We have generalised the observation that harmonic maps of $S^2$ to $CP^1$ and the Weierstrass system are related to the case of maps into $CP^{N-1}$ and of the corresponding generalised Weierstrass system. There is a difference; in the $CP^1$ case all harmonic maps are holomorphic (or antiholomorphic); in the $CP^{N-1}$ case, for $N > 2$, there are maps which are neither holomorphic nor antiholomorphic. Our generalised Weierstrass system misses out a part of the energy. To have full energy we need to go beyond the Weierstrass system and include also $\Phi$ variables.

It would be interesting to understand better how the existence of non-holomorphic harmonic maps translates into the properties of the Weierstrass system or its geometry. This problem is currently under investigation.

## References

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